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Optimization of Multilevel Investments Using Dynamic Programming Based on Fuzzy Cash Flows

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Abstract. Dynamic programming is applicable to any situation where items from several groups must be combined to form an entity, such as a composite investment or a transportation route connecting several districts. The most desirable entity is constructed in stages by forming sub-entities of progressively larger size. At each stage of the development, the sub-entities that are candidates for inclusion in the most desirable entity are retained, and all other sub-entities are discarded. In deterministic dynamic programming, a specification of the current state and current decision is enough to tell us with certainty the new state and costs during the current stage. In many practical problems, these factors may not be known with certainty, even if the current state and decision are known. In this paper, the dynamic programming is applied to the situation where each investment in the set has the following characteristics: the amount to be invested has several possible values, and the rate of return varies with the amount invested. Each sum that may be invested represents a distinct level of investment, and the investment therefore has multiple levels. A fuzzy present worth based dynamic programming approach is used. A numeric example for a multilevel investment with fuzzy geometric cash flows is given. A computer software named FUZDYN is developed for various problems such as alternatives having different lives, different uniform cash flows, and different ranking methods.

Keywords: fuzzy sets, dynamic programming, investment, fuzzy present worth

1. Introduction

Dynamic programming is a technique that can be used to solve many optimization problems. In most applications, dynamic programming obtains solutions by working backward from the end of a problem toward the beginning, thus breaking up a large, unwieldy problem into a series of smaller, more tractable problems. The characteristics of dynamic programming applications are (Winston (1994))

- The problem can be divided into stages with a decision required at each stage.
- Each stage has a number of states associated with it.
- The decision chosen at any stage describes how the state at the current stage is transformed into the state at the next stage.
- Given the current state, the optimal decision for each of the remaining stages must not depend on previously reached states or previously chosen decisions.

- If the states for the problem have been classified into one of T stages, there must be a recursion that relates the cost or reward earned during stages $t, t+1, \ldots, T$ to the cost or reward earned from stages $t+1, t+2, \ldots, T$.

The dynamic programming recursion can often be written in the following form. For a minimum problem with fixed output:

$$f_t(i) = \min\{(\textit{cost during stage } t) + f_{t+1}(\textit{new state at stage } t+1)\} \tag{1}$$

and for a maximum problem with fixed input, it is

$$f_t(i) = \max\{(\textit{benefits during stage } t) + f_{t+1}(\textit{new state at stage } t+1)\} \tag{2}$$

or for a maximum problem neither input nor output fixed, it is

$$f_t(i) = \max\{(\textit{`benefits} - \textit{costs' during stage } t) + f_{t+1}(\textit{new state at stage } t+1)\} \tag{3}$$

where the minimum in Eq. (1) or maximum in Eq. (2) and Eq. (3) is over all decisions that are allowable, or feasible, when the state at stage t is i. In Eq. (1), $f_t(i)$ is the minimum cost and in Eq. (2) the maximum benefit incurred from stage t to the end of the problem, given that at stage t the state is i.

In deterministic dynamic programming, a specification of the current state and current decision is enough to tell us with certainty the new state and costs during the current stage. In many practical problems, these factors may not be known with certainty, even if the current state and decision are known. When we use dynamic programming to solve problems in which the current period's cost or the next period's state is random, we call these problems *probabilistic dynamic programming problems (PDPs)*. In a *PDP*, the decision-maker's goal is usually to minimize the expected cost incurred or to maximize the expected reward earned over a given time horizon.

Many *PDPs* can be solved using recursions of the following forms. For minimum problems:

$$f_t(i) = \min_a \left\{ (\textit{expected cost during stage } t/i, a) + \sum_j p(j|i, a, t) f_{t+1}(j) \right\} \tag{4}$$

and for maximum problems:

$$f_t(i) = \max_a \left\{ (\textit{expected reward during stage } t/i, a) + \sum_j p(j|i, a, t) f_{t+1}(j) \right\} \tag{5}$$

where

i: the state at the beginning of stage t.
a: all actions that are feasible when the state at the beginning of stage t is i.
$p(j|i, a, t)$: the probability that the next period's state will be j, given that the current state is i and action a is chosen.

In the above formulations, we assume that benefits and costs received during later years are weighted the same as benefits and costs received during earlier years. But later benefits and costs should be weighted less than earlier benefits and costs. We can incorporate the time value of money into the dynamic programming recursion in the following way. For a maximum problem with neither input nor output fixed,

$$f_t(i) = max\left\{(\text{'}\mathit{benefits - costs}\text{'}\ \mathit{during\ stage}\ t) + \frac{1}{(1+r)} f_{t+1}(\mathit{new\ state\ at\ stage}\ t+1)\right\} \tag{6}$$

where r is the time value of money.

In most of the real-world problems, some of the decision data can be precisely assessed while others cannot. Humans are unsuccessful in making quantitative predictions, whereas they are comparatively efficient in qualitative forecasting. Furthermore, humans are more prone to interference from biasing tendencies if they are forced to provide numerical estimates since the elicitation of numerical estimates forces an individual to operate in a mode which requires more mental effort than that required for less precise verbal statements (Karwowski and Mital (1986)). Real numbers are used to represent data which can be precisely measured. For those data which cannot be precisely assessed, Zadeh's (1965) fuzzy sets can be used to denote them. The use of fuzzy set theory allows us to incorporate unquantifiable information, incomplete information, no obtainable information, and partially ignorant facts into the decision model. When decision data are precisely known, they should not be faced into a fuzzy format in the decision analysis.

Applications of fuzzy sets within the field of decision-making have, for the most part, consisted of extensions or "fuzzifications" of the classical theories of decision-making. While decision-making under conditions of risk and uncertainty have been modelled by probabilistic decision theories and by game theories, fuzzy decision theories attempt to deal with the vagueness or fuzziness inherent in subjective or imprecise determinations of preferences, constraints, and goals.

There are many imprecise and uncertain factors due to human's inherent subjectivity and vagueness in the articulation of their opinions. For an obvious reason, the analysis of multi-stage decision-making problems by conventional DP is rather difficult under fuzzy environments. Assuming that Zadeh's fuzzy sets theory was an appropriate way to deal with uncertainties and imprecision in real-world problems, DP was one of the earliest fundamental methodologies to which fuzzy sets theory was applied (Bellman and Zadeh (1970)), leading to what might be called fuzzy dynamic programming (FDP). FDP has received wide attention in many research and application fields during the last ten years.

Many capital budgeting problems allow of a *dynamic* formulation. There may actually be several decision points, but even if this is not so if the decision problem can be divided up into *stages* than a discrete dynamic expression is possible. Many problems allow of either static or dynamic expression. The choice of form would be up to the problem solver. Characteristically, a dynamic economizing model allocates scarce resources between alternative uses between initial and terminal times. In the case of equal-life multilevel investments, each investment in the set has the following characteristic: the amount to be invested has several *possible* values, and the rate of return varies with the amount invested. Each sum that may be invested represents a distinct *level* of investment, and the investment therefore has multiple levels. Examples of multilevel investments may be the purchase of labor-saving equipment where several types of equipment are available and each type has a unique cost. The level of investment in labor-saving equipment depends on the type of equipment selected. Another example is the construction and rental of an office building, where the owner-builder has a choice concerning the number of stories the building is to contain (Kurtz (1995)).

In the crisp optimization of multilevel investments using dynamic programming, cash flows are known with certainty. When cash flows cannot be precisely assessed, fuzzy numbers can be used to denote the estimations about them. Quite often in finance future cash amounts and interest rates are estimated. One usually employs educated guesses, based on expected values or other statistical techniques, to obtain future cash flows and interest rates. Statements like *approximately between $ 12,000 and $ 16,000* or *approximately between 10% and 15%* must be translated into an exact amount, such as *$ 14,000* or *12.5%* respectively. Appropriate fuzzy numbers can be used to capture the vagueness of those statements.

In the literature, fuzzy dynamic programming is classified as in Figure 1. The dynamic programming in this paper is based on a deterministic system under control with crisp termination time. But it is different from the approaches in Figure 1 because it is only based on fuzzy interval arithmetics.

This paper is organized as follows. First, the literature review on fuzzy dynamic programming is given. Then, dynamic programming algorithms based on crisp and fuzzy cash flows are explained. Finally, a numeric example for dynamic programming based on fuzzy cash flows is given. In the conclusion section, a comparison between DP based on crisp and fuzzy cash flows is made.

2. Literature Review

Fuzzy dynamic programming has found many applications to real-world problems: Health care, flexible manufacturing systems, integrated regional development, transportation networks and transportation of hazardous waste, chemical engineering, power and energy systems, and water resource systems.

Li and Lai (2001) developed a new fuzzy dynamic programming approach to solve hybrid multiobjective multistage decision-making problems. They presented a methodology of fuzzy evaluation and fuzzy optimization for hybrid multiobjective systems, in which the qualitative and quantitative objectives are synthetically considered. Esogbue (1999a) presented the essential elements of fuzzy dynamic programming and computational aspects as well as various key real world applications. Fu and Wang (1999) established a model in the framework of fuzzy project network by the team approach under the consideration of uncertain resource demand and the budget limit. The model is transformed into a classical linear program formula and its results show that the cause-effect relation of insufficient resources or over due of the project is identified for better

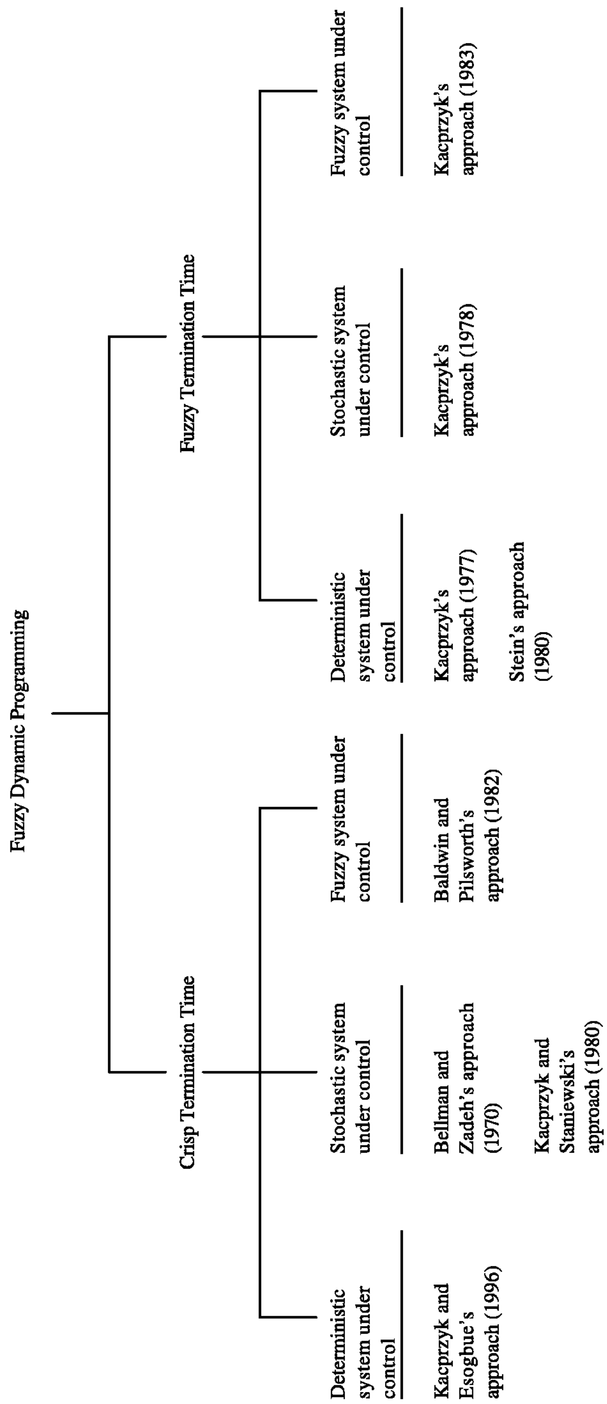


Figure 1. Classification of fuzzy dynamic programming.

management. Lai and Li (1999) developed another approach using dynamic programming to solve the multiple-objective resource allocation problem. There are two key issues being addressed in the approach. The first one is to develop a methodology of fuzzy evaluation and fuzzy optimization for multiple-objective systems. The second one is to design a dynamic optimization algorithm by incorporating the method of fuzzy evaluation and fuzzy optimization with the conventional dynamic programming technique. Esogbue (1999b) considered both time and space complexity problems associated with the fuzzy dynamic programming model. Huang et al (1998) developed a fuzzy dynamic programming approach to solve the direct bad control problem of the air conditioner loads. Kacprzyk and Esogbue (1996) did survey major developments and applications of fuzzy dynamic programming, which is advocated as a promising attempt at making dynamic programming models more realistic by a relaxation of often artificial assumptions of precision as to the constraints, goals, states and their transitions, termination time, etc. Chin (1995) proposed an approach using fuzzy dynamic programming to decide the optimal location and size of compensation shunt capacitors for distribution systems with harmonic distortion. The problem is formulated as a fuzzy dynamic programming of the minimization of real power loss and capacitor cost under the constraints of voltage limits and total harmonic distortion. Hussein and Abo-Sinna (1995) proposed an approach using fuzzy dynamic programming to solve the multiple criteria resource allocation problems. They concluded that solutions obtained by the approach are always efficient; hence an "optimal" compromise solution can be introduced. Berenji (1994) developed an algorithm called Fuzzy Q-Learning, which extends Watkin's Q-Learning method. It is used for decision processes in which the goals and/or the constraints, but not necessarily the system under control, are fuzzy in nature. He showed that fuzzy Q-Learning provides an alternative solution simpler than the Bellman-Zadeh's fuzzy dynamic programming approach.

3. Dynamic Programming Based on Crisp Cash Flows

Newnan (1988) showed that independent proposals competing for funding should be picked according to their IRR values- monotonically from highest to lowest. Ranking on present-worth values (computed at a specified MARR) may not give the same results. Given a specified minimum attractive rate of return (MARR) value, Newnan (1988) suggested that proposals be ranked on the basis of

$$\text{Ranking ratio} = \frac{\text{Proposal PW(MARR)}}{\text{Proposal first cost}} \tag{7}$$

where PW is the present worth of a proposal. The larger ratio indicates the better proposal.

Now assume that cash flows for l independent proposals that have passed a screening based on an MARR of r% are given in Table 1 and we have a \$$L$ capital limitation. The problem is which combination of proposals should be funded. The solution consists of the following steps:

Table 1. Cash flows for l independent proposals.

		End-of-period-cash-flow, $				
Proposal	Investment, $	Period 1	Period 2	Period 3	...	Period n
1	$X	CF_{11}^1	CF_{12}^1	CF_{13}^1	...	CF_{1n}^1
	$2X	CF_{11}^2	CF_{12}^2	CF_{13}^2	...	CF_{1n}^2
	$3X	CF_{11}^3	CF_{12}^3	CF_{13}^3	...	CF_{1n}^3
	...	...	...	...	...	...
	$$k$X	CF_{11}^k	CF_{12}^k	CF_{13}^k	...	CF_{1n}^k
2	$X	CF_{21}^1	CF_{22}^1	CF_{23}^1	...	CF_{2n}^1
	$2X	CF_{21}^2	CF_{22}^2	CF_{23}^2	...	CF_{2n}^2
	$3X	CF_{21}^3	CF_{22}^3	CF_{23}^3	...	CF_{2n}^3
	...	...	...	...	...	...
	$$k$X	CF_{21}^k	CF_{22}^k	CF_{23}^k	...	CF_{2n}^k
...	...	...	...	...	...	...
l	$X	CF_{l1}^1	CF_{l2}^1	CF_{l3}^1	...	$CF_{l\,n}^1$
	$2X	CF_{l1}^2	CF_{l2}^2	CF_{l3}^2	...	$CF_{l\,n}^2$
	$3X	CF_{l1}^3	CF_{l2}^3	CF_{l3}^3	...	$CF_{l\,n}^3$
	...	...	...	...	...	...
	$$k$X	CF_{l1}^k	CF_{l2}^k	CF_{l3}^k	...	$CF_{l\,n}^k$

1. Devise all possible investments that encompass proposals 1 and 2 alone, applying an upper limit of $$L$ to the amount invested. Compute the present worth of each proposal in the possible combinations using the discounted cash flow techniques. $$L$ can be allocated to proposal 1 alone or to proposal 2 alone or to any other combination.
2. Identify the most lucrative combination of proposals 1 and 2 corresponding to every possible value of $$L$, using the ranking ratio in Eq. (7).
3. Devise all possible investments that encompass proposals 1, 2, and 3, and identify the most lucrative one as in Step 2.
4. Continue increasing the number of proposals in the combination until the number is l and identify the most lucrative combination.

In Table 1, $CF_{l\,t}^k$ indicates the cash flow of proposal l in period t at the kth level of investment.

4. Dynamic Programming Based on Fuzzy Cash Flows

Given a fuzzy specified minimum attractive rate of return (MARR) value, proposals can be ranked on the basis of

$$\text{Ranking ratio} = \frac{\text{Proposal fuzzy PW(MARR)}}{\text{Proposal fuzzy first cost}} \tag{8}$$

where PW is the present worth of a proposal. The larger ratio indicates the better proposal.

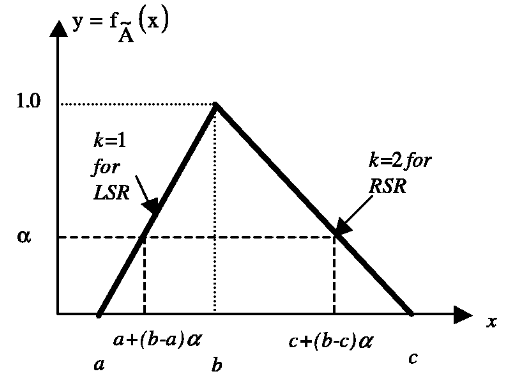


Figure 2. The membership function of a triangular fuzzy number, $\tilde{A} = (a, b, c)$.

It will be explained in the following how the fuzzy PW of a proposal can be calculated and how fuzzy numbers can be ranked. The extended algebraic operations of a triangular fuzzy numbers (TFN) can be found in (Kahraman (2001a), (2001b)).

It is necessary to use a ranking method to rank the fuzzy ranking ratios. In Section 4.2, some ranking methods are given. These methods use the forms of triangular or trapezoidal fuzzy numbers. The membership function of a *TFN*, $\tilde{A} = (a, b, c)$, is shown in Figure 2. *LSR* and *RSR* mean *"Left Side Representation"* and *"Right Side Representation"* respectively. $k = 1$ indicates *LSR* and $k = 2$ indicates *RSR*.

4.1. Fuzzy Present Worth (PW) Method

The present-worth method of alternative evaluation is very popular because future expenditures or receipts are transformed into equivalent dollars now. That is, all of the future cash flows associated with an alternative are converted into present dollars. If the alternatives have different lives, the alternatives must be compared over the same number of years.

Chiu and Park (1994) proposed a present worth formulation of a fuzzy cash flow. The result of the present worth is also a fuzzy number with a non-linear membership function.

$$P\tilde{W} = \left[\sum_{t=0}^{n}\left(\frac{\max(F_t^{l(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{r(y)})} + \frac{\min(F_t^{l(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{l(y)})}\right), \sum_{t=0}^{n}\left(\frac{\max(F_t^{r(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{l(y)})} + \frac{\min(F_t^{r(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{r(y)})}\right)\right] \tag{9}$$

where $F_t^{l(y)}$: the left representation of the cash at time t, $F_t^{r(y)}$: the right representation of the cash at time t, $r_t^{l(y)}$: the left representation of the interest rate at time t, $r_t^{r(y)}$: the right representation of the interest rate at time t.

Buckley's (1987) membership function for $P\widetilde{W}_n$,

$$\mu(x|P\widetilde{W}_n) = (pw_{n1}, f_{n1}(y|P\widetilde{W}_n)/pw_{n2}, pw_{n2}/f_{n2}(y|P\widetilde{W}_n), pw_{n3}) \quad (10)$$

is determined by

$$f_{ni}(y|P\widetilde{W}_n) = f_i(y|\widetilde{F})(1 + f_k(y|\widetilde{r}))^{-n} \quad (11)$$

for $i = 1, 2$ where $k = i$ for negative $\widetilde{F}$ and $k = 3 - i$ for positive $\widetilde{F}$.
Ward (1985) gave the fuzzy present worth function as

$$P\widetilde{W} = (1 + r)^{-n}(a, b, c, d) \quad (12)$$

where (a, b, c, d) is a flat fuzzy filter function (4F) number.

Kahraman et al (2000) and Kahraman (2001b) applied fuzzy present worth and fuzzy benefit/cost ratio analyses for the justification of manufacturing technologies and for public work projects. Karsak (1998) developed some measures of liquidity risk supplementing fuzzy discounted cash flow analysis. Boussabaine and Elhag (1999) examined the possible application of the fuzzy set theory to the cash flow analysis in construction projects. Dimitrovski and Matos (2000) presented an approach to including nonstatistical uncertainties in utility economic analysis by modeling uncertain variables with fuzzy numbers. Kuchta (2000) proposed fuzzy equivalents of all the classical capital budgeting methods.

When the value of a given cash flow differs from that of the previous cash flow by a constant percentage, *j*%, then the series is referred to as a *geometric series*. The present value of a crisp geometric series is given by

$$PW = \sum_{t=1}^{n} F_1(1 + g)^{t-1}(1 + i)^{-t} = \frac{F_1}{1 + g} \sum_{t=1}^{n} \left(\frac{1 + g}{1 + i}\right)^t \quad (13)$$

where F_1 is the first cash at the end of the first year. When this sum is made, the following present value equation is obtained:

$$PW = \begin{cases} F_1 \frac{1-(1+g)^n(1+i)^{-n}}{i-g}, & i \neq g \\ \frac{nF_1}{1+i}, & i = g \end{cases}. \quad (14)$$

In the case of fuzziness, the parameters used in Eq. (14) will be assumed to be fuzzy numbers, except project life. Let $\gamma(i, g, n) = \frac{1-(1+g)^n(1+i)^{-n}}{i-g}$, $i \neq g$. As it is in Figure 2, when $k = 1$, the left side representation will be depicted and when $k = 2$, the right side representation will be depicted. In this case, for $i \neq g$

$$f_{nk}(y|P\widetilde{W}_n) = f_k(y|\widetilde{F}_1)\gamma(f_{3-k}(y|\widetilde{i}),\ f_{3-k}(y|\widetilde{g}), n). \quad (15)$$

In Eq. (15), the least possible value is calculated for $k = 1$ and $y = 0$; the largest possible value is calculated for $k = 2$ and $y = 0$; the most promising value is calculated for $k = 1$ or $k = 2$ and $y = 1$ (Kahraman (2001a)).

4.2. Ranking Fuzzy Numbers

It is now necessary to use a ranking method to rank the TFNs such as Jain's (1976) method, Yager's (1980) method, Chang's (1981) method, Dubois and Prade's (1983) method, Kaufmann and Gupta's (1988) method, and Chiu and Park's (1994). For the ranking method of fuzzy variables, the main one is to employ expected value. The most general definition of expected value of a fuzzy variable and fuzzy expected value models are given by Liu and Liu (2002) and Liu (2002). These methods may give different ranking results and most methods are tedious in graphic manipulation requiring some complex mathematical calculation. In the following, three of the methods, which do not require graphical representations, are given.

Kaufmann and Gupta (1988) suggested three criteria for ranking *TFN*s with parameters (a, b, c). The dominance sequence is determined according to priority of:

1. Comparing the ordinary number $(a + 2b + c)/4$
2. Comparing the mode, (the corresponding most promise value), b, of each *TFN*.
3. Comparing the range, $c-a$, of each *TFN*.

The preference of projects is determined by the amount of their ordinary numbers. The project with the larger ordinary number is preferred. If the ordinary numbers are equal, the project with the larger corresponding most promising value is preferred. If projects have the same ordinary number and most promising value, the project with the larger range is preferred.

Liou and Wang (1992) proposed the total integral value method with an index of optimism $\omega \in [0, 1]$. Let $\widetilde{A}$ be a fuzzy number with a left membership function $f_{\widetilde{A}}^{L}$ and a right membership function $f_{\widetilde{A}}^{R}$. Then the total integral value is defined as:

$$E_{\omega}(\widetilde{A}) = \omega E_{R}(\widetilde{A}) + (1-\omega)E_{L}(\widetilde{A}) \tag{16}$$

where

$$E_{R}(\widetilde{A}) = \int_{\alpha}^{\beta} x f_{\widetilde{A}}^{R}(x)dx \tag{17}$$

$$E_{L}(\widetilde{A}) = \int_{\gamma}^{\delta} x f_{\widetilde{A}}^{L}(x)dx \tag{18}$$

where $-\infty \leq \alpha \leq \beta \leq \gamma \leq \delta \leq +\infty$ and a trapezoidal fuzzy number is denoted by $(\alpha, \beta, \gamma, \delta)$. For a triangular fuzzy number, $\widetilde{A} = (a, b, c)$,

$$E_{\omega}(\widetilde{A}) = \frac{1}{2}[\omega(a+b) + (1-\omega)(b+c)] \tag{19}$$

and for a trapezoidal fuzzy number, $\widetilde{B} = (\alpha, \beta, \gamma, \delta)$,

$$E_{\omega}(\widetilde{B}) = \frac{1}{2}[\omega(\gamma+\delta) + (1-\omega)(\alpha+\beta)]. \tag{20}$$

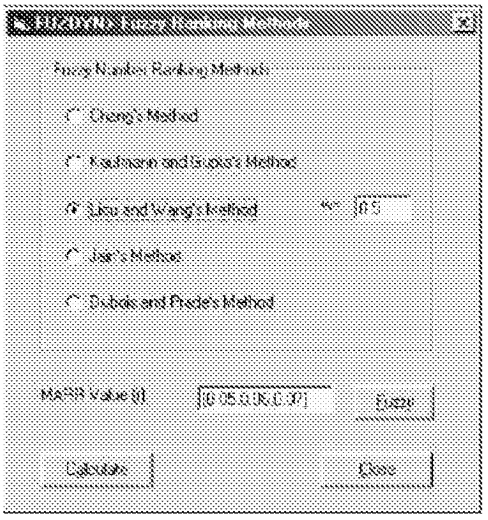


Figure 3. The form of fuzzy ranking methods.

Chiu and Park's (1994) weighted method for ranking *TFN*s with parameters (*a, b, c*) is formulated as

$$((a + b + c)/3) + wb$$

where w is a value determined by the nature and the magnitude of the most promising value. The preference of projects is determined by the magnitude of this sum.

The computer software developed by the authors, FUZDYN, has the ability to use many ranking methods which are tedious in graphic manipulation requiring some complex mathematical calculation. To select the ranking method required by the decision maker, the following form in Figure 3 is used:

4.3. Selection Among Equal-Life Multilevel Investments When Fuzzy Cash Flows Are Known

Now assume that cash flows for l independent proposals that have passed a screening based on an MARR of $\tilde{r}$% are given in Table 2 and we have a $\$\tilde{L}$ capital limitation. In Table 2, $\widetilde{CF}_{l\,t}^{k}$ indicates the fuzzy cash flow of proposal l in period t at the k th level of investment. The problem is which combination of proposals should be funded. The solution consists of the following steps:

1. Devise all possible investments that encompass proposals 1 and 2 alone, applying an upper limit of $\$\tilde{L}$ to the fuzzy amount invested. Compute the fuzzy present worth of

Table 2. Fuzzy cash flows for l independent proposals.

		End-of-period-cash-flow, \$				
Proposal	Investment, \$	Period 1	Period 2	Period 3	...	Period n
1	\$ $\tilde{X}$	$\widetilde{CF}^1_{11}$	$\widetilde{CF}^1_{12}$	$\widetilde{CF}^1_{13}$	...	$\widetilde{CF}^1_{1n}$
	2 $\tilde{X}$ \$	$\widetilde{CF}^2_{11}$	$\widetilde{CF}^2_{12}$	$\widetilde{CF}^2_{13}$	...	$\widetilde{CF}^2_{1n}$
	...	...	...	...	...	...
	k $\tilde{X}$ \$	$\widetilde{CF}^k_{11}$	$\widetilde{CF}^k_{12}$	$\widetilde{CF}^k_{13}$		$\widetilde{CF}^k_{1n}$
2	\$ $\tilde{X}$	$\widetilde{CF}^1_{21}$	$\widetilde{CF}^1_{22}$	$\widetilde{CF}^1_{23}$	...	$\widetilde{CF}^1_{2n}$
	2 $\tilde{X}$ \$	$\widetilde{CF}^2_{21}$	$\widetilde{CF}^2_{22}$	$\widetilde{CF}^2_{23}$	...	$\widetilde{CF}^2_{2n}$
	...	...	...	...	...	...
	k $\tilde{X}$ \$	$\widetilde{CF}^k_{21}$	$\widetilde{CF}^k_{22}$	$\widetilde{CF}^k_{23}$	...	$\widetilde{CF}^k_{2n}$
...	...	...	...	...	...	...
l	\$ $\tilde{X}$	$\widetilde{CF}^1_{l1}$	$\widetilde{CF}^1_{l2}$	$\widetilde{CF}^1_{l3}$	...	$\widetilde{CF}^1_{l\,n}$
	2 $\tilde{X}$ \$	$\widetilde{CF}^2_{l1}$	$\widetilde{CF}^2_{l2}$	$\widetilde{CF}^2_{l3}$	...	$\widetilde{CF}^2_{l\,n}$
...	...	...	...	...	...	...
	k $\tilde{X}$ \$	$\widetilde{CF}^k_{l1}$	$\widetilde{CF}^k_{l2}$	$\widetilde{CF}^k_{l3}$	...	$\widetilde{CF}^k_{l\,n}$

each proposal in the possible combinations using the fuzzy discounted cash flow techniques (Kahraman et al (2002)). $\$\tilde{L}$ can be allocated to proposal 1 alone or to proposal 2 alone or to any other combination.

2. Identify the most lucrative combination of proposals 1 and 2 corresponding to every possible value of $\$\tilde{L}$, using the ranking ratio in Eq. (8). Use a ranking method of fuzzy numbers to identify the most lucrative combination.
3. Devise all possible investments that encompass proposals 1, 2, and 3, and identify the most lucrative one as in Step 2. Use a ranking method of fuzzy numbers to identify the most lucrative combination.
4. Continue increasing the number of proposals in the combination until the number is l and identify the most lucrative combination. Use a ranking method of fuzzy numbers to identify the most lucrative combination.

4.4. A Numerical Example

A firm has \$(15000, 21000, 27000) available for investment, and three investment proposals are under consideration. Each proposal has these features: the amount that can be invested is a multiple of \$(5000, 7000, 9000); the investors receive annual unequal receipts; each proposal has a useful life of three years. Table 3 lists the annual geometric receipts corresponding to the various fuzzy levels of investment. Devise the most lucrative composite investment using fuzzy dynamic programming. The company-specified MARR value $\tilde{r}\%$ is (5%, 6%, 7%) per year.

In FUZDYN, the project definition is as in Figure 4.

As it can be seen from Table 3, the geometric growth rates (g) for the annual receipts at the investment levels are 10%, 12%, and 14% respectively and they are given as crisp rates in the problem. Using Eq. (15), $f_1\left(y\middle|\tilde{r}\right) = 0.05 + 0.01y,\ \ f_2\left(\middle|\tilde{r}\right) = 0.07 - 0.01y, \gamma$ $\left(f_{3-k}\left(y\middle|\tilde{r}\right), g, n\right), k = 1, 2.$

Table 3. Fuzzy cash flows for three independent proposals.

Proposal	Investment, $	Year 1	Year 2	Year 3
1	$(5000, 7000, 9000)	(3000, 4000, 5000)	(3300, 4400, 5500)	(3630, 4840, 6050)
	$(10000, 14000, 18000)	(5000, 6000, 7000)	(5600, 6720, 7840)	(6272, 7526, 8780)
	$(15000, 21000, 27000)	(8000, 9000, 10000)	(9120, 10260, 11400)	(10396, 11696, 12996)
2	$(5000, 7000, 9000)	(3000, 4000, 6000)	(3300, 4400, 6600)	(3630, 4840, 7392)
	$(10000, 14000, 18000)	(4000, 6000, 7000)	(4480, 6720, 7840)	(5017, 7526, 8780)
	$(15000, 21000, 27000)	(5000, 9000, 10000)	(5700, 10260, 11400)	(6498, 11696, 12996)
3	$(5000, 7000, 9000)	(3000, 3000, 4000)	(3300, 3300, 4400)	(3630, 3630, 4840)
	$(10000, 14000, 18000)	(5000, 7000, 7000)	(5600, 7840, 7840)	(6272, 7526, 7526)
	$(15000, 21000, 27000)	(8000, 9000, 12000)	(9120, 10260, 13680)	(10396, 11696, 15595)

In FUZDYN, data input for proposals is shown in Figure 5. In Figure 6a, the data regarding fuzzy investment cost, fuzzy growth rate, and the benefit of the first year are entered and in Figure 6b, it is shown how a fuzzy number is entered.
For the total investment of $(15000, 21000, 27000) in proposals 1 and 2:

Investment in proposal 1: $ (15000, 21000, 27000) and proposal 2: $ 0

We find $f_1(y|\widetilde{F}_1) = 1000y + 8000,\ f_2(y|\widetilde{F}_1) = 10000 - 1000y$.

For $k = 1, \mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = (1000y + 8000)\left[\dfrac{(1.14)^3(1.07 - 0.01y)^{-3} - 1}{0.07 + 0.01y}\right]$ and for $y = 0$, $\mathrm{f}_{3,1}\left(\mathrm{y}\middle|\mathrm{P}\widetilde{\mathrm{W}}\right) =$ $ 23,929 and for $y = 1, \mathrm{f}_{3,1}\left(\mathrm{y}\middle|\mathrm{P}\widetilde{\mathrm{W}}\right) =$ $ 27,442. For $k = 2, \mathrm{f}_{3,2}\left(\mathrm{y}\middle|\mathrm{P}\widetilde{\mathrm{W}}\right) =$ $(10000 - 1000\,\mathrm{y})\left[\dfrac{1 - (1.14)^3(1.05 + 0.01\,\mathrm{y})^{-3}}{0.01\mathrm{y} - 0.09}\right]$ and for $y = 0, \mathrm{f}_{3,2}\left(\mathrm{y}\middle|\mathrm{P}\widetilde{\mathrm{W}}\right) =$ $ 31,090.

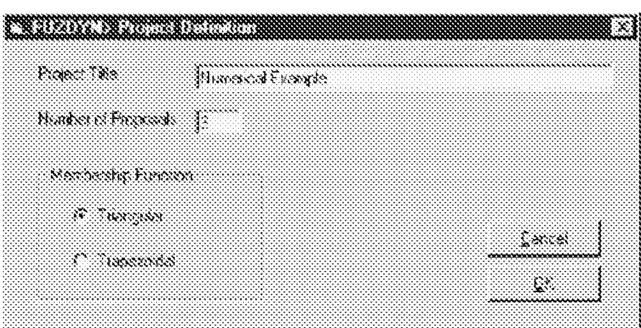


Figure 4. Project definition.

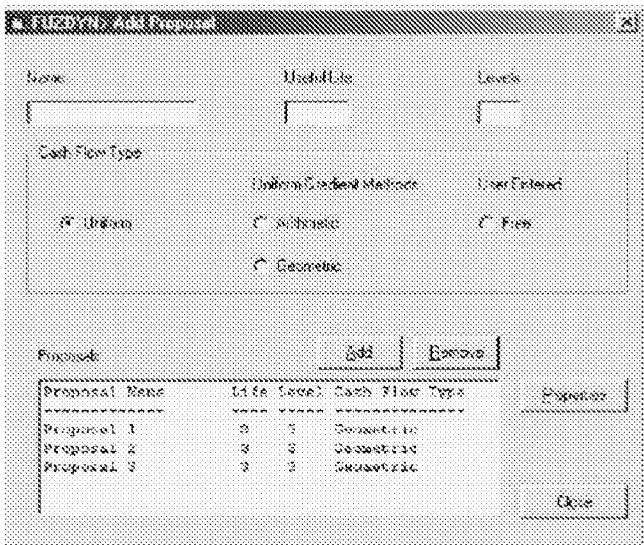
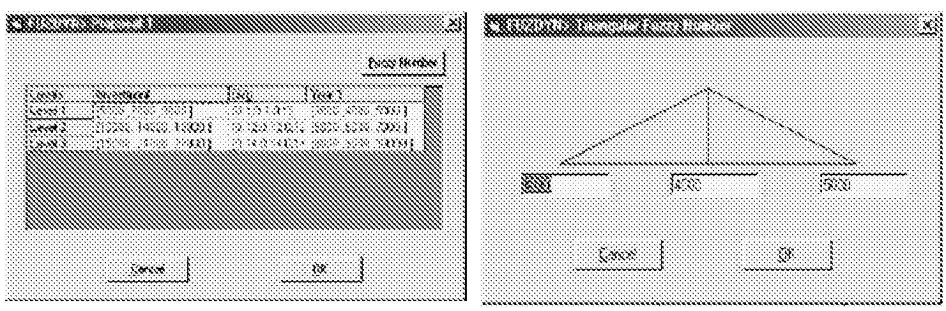


Figure 5. The form of parameter input for proposals.



(a) (b)

Figure 6. The forms related to data input for proposals.

Now we can calculate the net $\mathrm{P\widetilde{W}}$ and the fuzzy ranking ratio:

$$\mathrm{N\widetilde{P}W}_1 = \$(23{,}929; 27{,}442; 31{,}090) - \$(15{,}000; 21{,}000; 27{,}000)$$
$$= \$(-3{,}071; +6{,}442; +16{,}090)$$

$$\text{Ranking ratio} = \frac{\$(-3{,}071; +6{,}442; +16{,}090)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.114; +0{,}307; +1{,}073)$$

Investment in proposal 1: $ (10000, 14000, 18000) and proposal 2: $ (5000, 7000, 9000)

For proposal 1:

$$f_1(y|\widetilde{F}_1) = 1000\,y + 5000,\ f_2(y|\widetilde{F}_1) = 7000 - 1000\,y.$$

For $k = 1, \mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = (1000y + 5000)\left[\dfrac{(1.12)^3(1.07 - 0.01y)^{-3} - 1}{0.05 + 0.01y}\right]$ and for $y = 0$, $\mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = \$\ 14{,}684$ and for $y = 1, \mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = \$\ 17{,}960$. For $k = 2, \mathrm{f}_{3,2}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = (7000 - 1000\,y)\left[\dfrac{1 - (1.12)^3(1.05 + 0.01y)^{-3}}{0.01y - 0.07}\right]$ and for $y = 0, \mathrm{f}_{3,2}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = \$\ 21{,}363$.

For proposal 2:

$$f_1(y|\widetilde{F}_1) = 1000\,y + 3000,\ f_2(y|\widetilde{F}_1) = 6000 - 2000\,y$$

For $k = 1, \mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = (1000y + 3000)\left[\dfrac{(1.10)^3(1.07 - 0.01y)^{-3} - 1}{0.03 + 0.01y}\right]$ and for $y = 0$, $\mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = \$\ 8{,}649$ and for $y = 1, \mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = \$\ 11{,}753$. For $k = 2, \mathrm{f}_{3,2}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = (6000 - 2000\,\mathrm{y})\left[\dfrac{1 - (1.10)^3(1.05 + 0.01\mathrm{y})^{-3}}{0.01\mathrm{y} - 0.05}\right]$ and for $\mathrm{y} = 0$, $\mathrm{f}_{3,2}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = \$17{,}972$.

Now we can calculate the net $\mathrm{P}\widetilde{\mathrm{W}}$ and the fuzzy ranking ratio:

$$\begin{aligned}\mathrm{P}\widetilde{\mathrm{W}}_{1,2} &= \mathrm{P}\widetilde{\mathrm{W}}_1 + \mathrm{P}\widetilde{\mathrm{W}}_2 = \$(14{,}684; 17{,}690; 21{,}393) + \$(8{,}649; 11{,}753; 17{,}972)\\ &= \$(23{,}333; 29{,}443; 39{,}365)\end{aligned}$$

$$\begin{aligned}\mathrm{N}\widetilde{\mathrm{P}}\mathrm{W}_{1,2} &= \$(23{,}333; 29{,}443; 39{,}365) - \$(15{,}000; 21{,}000; 27{,}000)\\ &= \$(-3{,}667; +8{,}443; +24{,}365)\end{aligned}$$

$$\textit{Ranking ratio} = \frac{\$(-3{,}667; +8{,}443; +24{,}365)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.136; +0.402; +1.624)$$

Investment in proposal 1: \$ (5000, 7000, 9000) and proposal 2: \$ (10000, 14000, 18000)

For proposal 1:

$$f_1(y|\widetilde{F}_1) = 1000\,y + 3000,\ f_2(y|\widetilde{F}_1) = 5000 - 1000\,y.$$

For $k = 1, \mathrm{f}_{3,1}(\mathrm{y}|\mathrm{P}\widetilde{\mathrm{W}}) = (1000y + 3000)\left[\dfrac{(1.10)^3(1.07 - 0.01y)^{-3} - 1}{0.03 + 0.01y}\right]$ and for $y = 0$,

$f_{3,1}(y|\tilde{PW}) = \$ \ 8{,}649$ and for $y = 1$, $f_{3,1}(y|\tilde{PW}) = \$ \ 11{,}753$. For $k = 2$, $f_{3,2}(y|\tilde{PW}) = (5000 - 1000\,y)\left[\frac{1-(1.10)^3(1.05+0.01y)^{-3}}{0.01y-0.05}\right]$ and for $y = 0$, $f_{3,2}(y|\tilde{PW}) = \$ \ 14{,}977$.

For proposal 2:

$f_1(y|\tilde{F}_1) = 2000\,y + 4000$, $f_2(y|\tilde{F}_1) = 7000 - 1000\,y$

For $k = 1$, $f_{3,1}(y|\tilde{PW}) = (2000y + 4000)\left[\frac{(1.12)^3(1.07-0.01y)^{-3}-1}{0.05+0.01y}\right]$ and for $y = 0$, $f_{3,1}(y|\tilde{PW}) = \$ \ 11{,}747$ and for $y = 1$, $f_{3,1}(y|\tilde{PW}) = \$ \ 17{,}960$. For $k = 2$, $f_{3,2}(y|\tilde{PW}) = (7000 - 1000\,y)\left[\frac{1-(1.12)^3(1.05+0.01y)^{-3}}{0.01y-0.07}\right]$ and for $y = 0$, $f_{3,2}(y|\tilde{PW}) = \$ \ 21{,}363$.

Now we can calculate the net PW and the fuzzy ranking ratio:

$$\tilde{PW}_{1,2} = \tilde{PW}_1 + \tilde{PW}_2 = \$(8{,}649; 11{,}753; 14{,}977) + \$(11{,}747; 17{,}960; 21{,}363)$$
$$= \$(20{,}396; 29{,}713; 36{,}340)$$

$$N\tilde{P}W_{1,2} = \$(20{,}396; 29{,}713; 36{,}340) - \$(15{,}000; 21{,}000; 27{,}000)$$
$$= \$(-6{,}604; +8{,}713; +21{,}340)$$

$$\textit{Ranking ratio} = \frac{\$(-6{,}604; +8{,}713; +21{,}340)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.025; +0.415; +1.423)$$

Investment in proposal 1: \$ 0 and proposal 2: \$ (15000, 21000, 27000)

We find $f_1(y|\tilde{F}_1) = 4000\,y + 5000$, $f_2(y|\tilde{F}_1) = 10000 - 1000\,y$.

For $k = 1$, $f_{3,1}(y|\tilde{PW}) = (4000y + 5000)\left[\frac{(1.14)^3(1.07-0.01y)^{-3}-1}{0.07+0.01y}\right]$ and for $y = 0$, $f_{3,1}(y|\tilde{PW}) = \$ \ 14{,}956$ and for $y = 1$, $f_{3,1}(y|\tilde{PW}) = \$ \ 27{,}442$. For $k = 2$, $f_{3,2}(y|\tilde{PW}) = (10000 - 1000\,y)\left[\frac{1-(1.14)^3(1.05+0.01\,y)^{-3}}{0.01y-0.09}\right]$ and for $y = 0$, $f_{3,2}(y|\tilde{PW}) = \$ \ 31{,}090$.

Now we can calculate the net PW and the fuzzy ranking ratio:

$$N\tilde{P}W_2 = \$(14{,}956; 27{,}442; 31{,}090) - \$(15{,}000; 21{,}000; 27{,}000)$$
$$= \$(-12{,}044; +6{,}442; +16{,}090)$$

Table 4. Identifying the most lucrative combination of $ (15,000; 21,000; 27,000) for the First Stage.

Ranking ratio, $\widetilde{A}$	$E_\omega(\widetilde{A}) = \frac{1}{2}[\omega(a+b) + (1-\omega)(b+c)]$
(−0.114; +0.307; +1.073)	0.393
(−0.136; +0.402; +1.624)	0.573*
(−0.025; +0.415; +1.423)	0.557
(−0.446; +0.307; +1.073)	0.310

$$Ranking\ ratio = \frac{\$(-12{,}044; +6{,}442; +16{,}090)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.446; +0{,}307; +1{,}073)$$

To select the most lucrative combination of an investment of $ (15,000; 21,000; 27,000), we will use Liou and Wang's (1992) method. For a moderately optimistic decision-maker, $\omega = 0.5$.

As it can be seen from Table 4, the most lucrative combination is to invest $ (10,000; 14,000; 18,000) in proposal 1 and invest $ (5,000; 7,000; 9,000) in proposal 2.

For the total investment of $ (10000, 14000, 18000) in proposals 1 and 2:

Investment in proposal 1: (10000, 14000, 18000) and proposal 2: $ 0

$$Ranking\ ratio = \frac{\$(-3{,}316; +3{,}960; +11{,}363)}{\$(10{,}000; 14{,}000; 18{,}000)} = (-0.184; +0.283; +1.136)$$

Investment in proposal 1: $ (5000, 7000, 9000) and proposal 2: $ (5000, 7000, 9000)

$$Ranking\ ratio = \frac{\$(-702; +9{,}506; +22{,}949)}{\$(10{,}000; 14{,}000; 18{,}000)} = (-0.039; +0.679; +2.295)$$

Investment in proposal 1: 0 $ and proposal 2: $ (10000, 14000, 18000)

$$Ranking\ ratio = \frac{\$(-6{,}253; +3{,}960; +11{,}363)}{\$(10{,}000; 14{,}000; 18{,}000)} = (-0.347; +0.283; +1.136)$$

To select the most lucrative combination of an investment of $ (10,000; 14,000; 18,000), we will again use Liou and Wang's (1992) method. For a moderately optimistic decision-maker, $\omega = 0.5$.

As it can be seen from Table 5, the most lucrative combination is to invest $ (5,000; 7,000; 9,000) in proposal 1 and invest $ (5,000; 7,000; 9,000) in proposal 2.

For the total investment of $ (5000, 7000, 9000) in proposals 1 and 2:

Table 5. Identifying the most lucrative combination of \$ (10,000; 14,000; 18,000) for the Second Stage.

Ranking ratio, $\widetilde{A}$	$E_\omega(\widetilde{A}) = \frac{1}{2}[\omega(a+b)+(1-\omega)(b+c)]$
(−0.184; +0.282; +1.136)	0.379
(−0.039; +0.679; +2.295)	0.904*
(−0.347; +0.283; +1.136)	0.339

Table 6. Identifying the most lucrative combination of \$ (15,000; 21,000; 27,000) for the Last Stage.

Ranking ratio, $\widetilde{A}$	$E_\omega(\widetilde{A}) = \frac{1}{2}[\omega(a+b)+(1-\omega)(b+c)]$
(−0.136; +0.402; +1.624)	0.573
(−0.039; +0.539; +1.995)	0.759*
(−0.136; +0.557; +1.622)	0.650
(−0.114; +0.307; +0.687)	0.328

Investment in proposal 1: \$ (5000, 7000, 9000) and proposal 2: \$ 0

$$Ranking\ ratio = \frac{\$(-351; +4{,}753; +9{,}977)}{\$(5{,}000; 7{,}000; 9{,}000)} = (-0.039; +0.679; +1.995)$$

Investment in proposal 1: 0 and proposal 2: \$ (5000, 7000, 9000)

$$Ranking\ ratio = \frac{\$(-351; +4{,}753; +12{,}972)}{\$(5{,}000; 7{,}000; 9{,}000)} = (-0.039; +0.679; +2.594)$$

It is obvious that the most lucrative combination of an investment of \$ (5,000; 7,000; 9,000) is to invest \$ (5,000; 7,000; 9,000) in proposal 2.

Now we will devise all possible investments that encompass proposals 1, 2, and 3, and identify the most lucrative one.

- Investment in proposals 1 + 2: \$ (15000, 21000, 27000) and proposal 3: \$ 0

$$\text{Ranking ratio} = \frac{\$(-3{,}667; +8{,}443; +24{,}365)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.136; +0.402; +1.624)$$

- Investment in proposals 1 + 2: \$ (10000, 14000, 18000) and proposal 3: \$ (5000, 7000, 9000)

$$\text{Ranking ratio} = \frac{\$(-1{,}053; +11{,}321; +29{,}930)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.039; +0.539; +1.995)$$

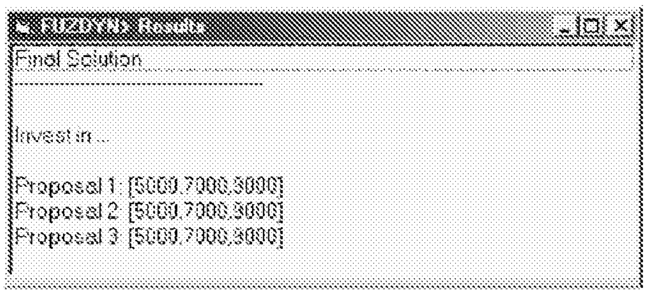


Figure 7. Final solution.

- Investment in proposals 1 + 2: \$ (5000, 7000, 9000) and proposal 3: \$ (10000, 14000, 18000)

$$\textit{Ranking ratio} = \frac{\$(-3{,}667; +11{,}707; +24{,}335)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.136; +0.557; +1.622)$$

- Investment in proposals 1 + 2: \$ 0 and proposal 3: \$ (15000, 21000, 27000)

$$\textit{Ranking ratio} = \frac{\$(-3{,}071; +6{,}442; +10{,}308)}{\$(15{,}000; 21{,}000; 27{,}000)} = (-0.114; +0.307; +0.687)$$

To select the most lucrative combination of an investment of \$ (15,000; 21,000; 27,000), we will again use Liou and Wang's (1992) method. For a moderately optimistic decision-maker, $\omega = 0.5$.

As it can be seen from Table 6, the most lucrative combination is to invest \$ (10,000; 14,000; 18,000) in proposal 1 and proposal 2 and invest \$ (5,000; 7,000; 9,000) in proposal 3. Then the final solution is to invest \$ (5000, 7000, 9000) in proposal 1 and \$ (5000, 7000, 9000) in proposal 2, and \$ (5,000; 7,000; 9,000) in proposal 3.

The final solution found by FUZDYN is given in Figure 7:

When the problem is solved in the crisp case, at the end of the first stage, the largest ranking ratio will be 0.415 and the most lucrative combination is to invest \$ 7000 in proposal 1 and \$ 14000 in proposal 2. At the end of the second stage, the largest ranking ratio will again be 0.679 and the most lucrative combination is to invest \$ 7000 in proposal

Table 7. The ranking ratios for the last stage in the crisp case.

Combination No	Investment	Ranking ratios
1	\$ 21000 in proposal 1 + 2 and \$ 0 in proposal 3	0.415
2	\$ 14000 in proposal 1 + 2 and \$ 7000 in proposal 3	0.539
3	\$ 7000 in proposal 1, \$ 0 in proposal 2 and \$ 14000 in proposal 3	0.557
4	\$ 7000 in proposal 2, \$ 0 in proposal 0 and \$ 14000 in proposal 3	0.557
5	\$ 0 in proposal 1 + 2 and \$ 21000 in proposal 3	0.307

1 and $ 7000 in proposal 2. At the end of the third stage, the largest ranking ratios are the same for both proposals and the decision-maker is free to select any of the proposals. The ranking ratios and combinations for the last stage are given in Table 7.

The most lucrative one is either of third or fourth combinations. Because the ranking method takes care of the least and the largest possible values, the proposals selected in the fuzzy case are different the ones in the crisp case.

5. Conclusions

Dynamic programming is a powerful optimization technique that is particularly applicable to many complex problems requiring a sequence of interrelated decisions. In this paper, we presented a fuzzy dynamic programming application for the selection of independent multilevel investments. This method should be used when imprecise or fuzzy input data or parameters exist. In multilevel mathematical programming, input data or parameters are often imprecise or fuzzy in a wide variety of hierarchical optimization problems such as defense problems, transportation network designs, economical analysis, financial control, energy planning, government regulation, equipment scheduling, organizational management, quality assurance, conflict resolution and so on. Developing methodologies and new concepts for solving fuzzy and possibilistic multilevel programming problems is a practical and interesting direction for future studies.

Notes

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